



Variational aspects of the geodesics problem in sub-Riemannian geometry[☆]

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Abstract

We study the local geometry of the space of horizontal curves with endpoints freely varying in two given submanifolds \mathcal{P} and \mathcal{Q} of a manifold \mathcal{M} endowed with a distribution $\mathcal{D} \subset T\mathcal{M}$. We give a different proof, that holds in a more general context, of a result by Bismut [Large Deviations and the Malliavin Calculus, Progress in Mathematics, Birkhauser, Boston, 1984, Theorem 1.17] stating that the normal extremizers that are not abnormal are critical points of the sub-Riemannian action functional. We use the Lagrangian multipliers method in a Hilbert manifold setting, which leads to a characterization of the abnormal extremizers (critical points of the endpoint map) as curves where the linear constraint fails to be regular. Finally, we describe a modification of a result by Liu and Sussmann [Memoirs Am. Math. Soc. 564 (1995) 118] that shows the global distance minimizing property of sufficiently small portions of normal extremizers between a point and a submanifold. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

A sub-Riemannian manifold consists of a smooth n -dimensional manifold \mathcal{M} , and a smooth distribution $\mathcal{D} \subset T\mathcal{M}$ on \mathcal{M} of constant rank $n - k$, endowed with a smoothly

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varying positive definite metric tensor g . The length is defined only for *horizontal* curves in \mathcal{M} , i.e., curves which are everywhere tangent to \mathcal{D} . It was proven in [9] that a horizontal curve which minimizes length is either a *normal extremal* or an *abnormal extremal*, where the two possibilities are not mutually exclusive. This proof is obtained as an application of the *Pontryagin maximum principle* of Optimal Control Theory; an alternative proof of this fact obtained by variational methods is given in this paper (Corollary 5.8).

A normal extremal is defined as a curve in \mathcal{M} that is a solution of the sub-Riemannian Hamiltonian $H(p) = \frac{1}{2}g^{-1}(p|_{\mathcal{D}}, p|_{\mathcal{D}})$ on $T\mathcal{M}^*$, i.e., a curve that is the projection on \mathcal{M} of an integral line of the Hamiltonian flow \vec{H} . Such curves are automatically horizontal. An abnormal extremal can be defined as a curve which is the projection on \mathcal{M} of a non-zero *characteristic curve* in the annihilator $\mathcal{D}^0 \subset T\mathcal{M}^*$; a characteristic curve is a curve in \mathcal{D}^0 which is tangent to the kernel of the restriction to \mathcal{D}^0 of the canonical symplectic form of $T\mathcal{M}^*$.

As in the case of Riemannian geodesics, sufficiently small segments of a normal extremal is length minimizing (see [9]); however, “most” abnormal extremals do not have any sort of minimizing property (observe that the definition of abnormal minimizer does not involve the metric g).

The first example of a length minimizer which is *not* a normal extremal was given in [11]. The goal of this paper is to discuss the theory of extremals by techniques of Calculus of Variations and to give the basic instruments to develop a variational theory (Morse theory, Ljusternik–Schnirelman theory) for sub-Riemannian geodesics. The results of this paper are used in [4], where the authors consider the problem of existence and multiplicity of geodesics joining a point and a line in a sub-Riemannian manifold $(\mathcal{M}, \mathcal{D}, g)$, with $\text{codim}(\mathcal{D}) = 1$.

In [2, Theorem 1.17] it is proven that the normal sub-Riemannian extremals between two fixed points of a sub-Riemannian manifold are critical points of the sub-Riemannian action functional. The proof is presented in the context of the *Malliavin calculus*, employed to study some problems connected with the asymptotics of the semi-group associated with a hypoelliptic diffusion. For this purposes, the author’s proof is restricted to the case that the image of the normal extremal be contained in an open subset of \mathcal{M} on which the distribution \mathcal{D} is globally generated by $n - k$ smooth vector fields. In this paper, we reprove the result of Bismut [2, Theorem 1.17] under the more general assumptions that:

- the vector bundle \mathcal{D} is not necessarily trivial around the image of the normal extremizer;
- the endpoints of the normal extremizers are free to move on two submanifolds of \mathcal{M} .

As to the first generalization of the extremizing property of the normal extremizers, it is interesting to observe that in the proof it is employed the Lagrangian multipliers technique that uses *time-dependent referentials* of \mathcal{D} defined in a neighborhood of the graph of any continuous curve in \mathcal{M} . The existence of such referentials is obtained by techniques of calculus with affine connections, and it is likely that the method of time-dependent referentials may be applied to other situations where global geometrical results are to be proven. For instance, in [7] the author proves a Morse index theorem for normal extremizers, but in his proof he implicitly assumes the triviality of the vector bundle \mathcal{D} in a neighborhood of the

curve. However, the arguments presented could be made more precise by a systematic use of time-dependent referentials.

Another observation that is worth making about the Lagrangian multipliers is that, in the functional setup of the method, the constraint is given by the kernel of a suitable submersion (see formula (3)) from the set of H^1 -curves in an open subset of \mathcal{M} taking values in the Hilbert space of \mathbb{R}^k -valued L^2 -functions. This submersion is defined using time-dependent referentials of the annihilator \mathcal{D}^0 of \mathcal{D} in the cotangent bundle $T\mathcal{M}^*$, and the surprising result is that such map fails to be a submersion precisely at the abnormal extremizers. We therefore obtain a new variational description of the abnormal extremizers in a sub-Riemannian manifold.

Finally, it is important to emphasize the role of the endmanifolds \mathcal{P} and \mathcal{Q} in the development of the theory. An interesting result is that, if either one of the two is everywhere transversal to \mathcal{D} , then the set of horizontal curves between \mathcal{P} and \mathcal{Q} does not contain singularities (Proposition 5.4); in particular, all the sub-Riemannian extremizers between \mathcal{P} and \mathcal{Q} are normal. This fact can be used in several circumstances: for instance, in Corollary 5.6, we obtain some information about the geometry of sub-Riemannian balls; moreover, it is possible to obtain also some criteria to establish the smoothness for abnormal extremizers (see Remark 5.7).

We outline briefly the contents of each section of this article.

In Section 2, we study the local geometry of the space of horizontal curves joining two fixed points q_0 and q_1 of \mathcal{M} by two different techniques. On one hand, this space can be described as the set of curves γ joining q_0 and q_1 satisfying $\theta_i(\dot{\gamma}) = 0$, where $\theta_1, \dots, \theta_k$ is a local time-dependent referential for the annihilator \mathcal{D}^0 of \mathcal{D} . On the other hand, the same space can be obtained as the inverse image of q_1 by the *endpoint mapping* restricted to the set of horizontal curves emanating from q_0 (see [10]). We show that these two constraints have the same regular points; such curves are called *regular* and a suitable neighborhood of them in the space of horizontal curves joining q_0 and q_1 has the structure of an infinite dimensional Hilbert manifold.

In Section 3, we define the normal extremals, also called *normal geodesics*, in a sub-Riemannian manifold, using the Hamiltonian setup.

In Section 4, we study the image of the differential of the endpoint mapping; to this aim we introduce an atlas on the space of horizontal curves starting at q_0 .

Finally, in Section 5, we prove that a regular curve is a critical point of the sub-Riemannian action functional if and only if it is a normal geodesic. We also study the case of curves with endpoints varying in two submanifolds of \mathcal{M} . If we consider the space of horizontal curves joining the submanifolds \mathcal{P} and \mathcal{Q} , then provided that either \mathcal{P} or \mathcal{Q} is transversal to \mathcal{D} , this set is always a Hilbert manifold. Moreover, the critical points of the sub-Riemannian action functional in this space are those normal geodesics between \mathcal{P} and \mathcal{Q} whose Hamiltonian lift annihilates the tangent spaces of \mathcal{P} and \mathcal{Q} at its endpoints.

To conclude the paper, we present two short appendices. In Appendix A, we prove that every horizontal curve can be obtained as the reparameterization of an affinely parameterized horizontal curve. In Appendix B, we adapt a proof of local optimality of normal geodesics

due to [9, Appendix C] to prove that sufficiently small portions of normal geodesics are length minimizers between an initial submanifold and a point.

2. The differentiable structure of the space of horizontal curves

We give a couple of preliminary results needed to the study of the geometry of the set of horizontal paths in a sub-Riemannian manifold. The main reference for the geometry of infinite dimensional manifolds is [8]; for the basics of Riemannian geometry, we refer to [3].

Recall that a smooth map $f : M \mapsto N$ between Hilbert manifolds is a *submersion* at $x \in M$ if the differential $df(x) : T_x M \mapsto T_{f(x)} N$ is surjective; f is a submersion if it is a submersion at every $x \in M$.

Lemma 2.1. *Let M, M_1 and M_2 be Hilbert manifolds and let $f : M \mapsto M_1, g : M \mapsto M_2$ be submersions. Let $p_1 \in M_1, p_2 \in M_2$ and choose $x \in f^{-1}(p_1) \cap g^{-1}(p_2)$. Then, $f|_{g^{-1}(p_2)}$ is a submersion at x , if and only if $g|_{f^{-1}(p_1)}$ is a submersion at x .*

Proof. We need to show that $df(x)|_{\text{Ker}(dg(x))}$ is surjective onto $T_{f(x)} M_1$ if and only if $dg(x)|_{\text{Ker}(df(x))}$ is surjective onto $T_{g(x)} M_2$. This follows from a general fact: if $T : V \mapsto V_1$ and $S : V \mapsto V_2$ are surjective linear maps between vector spaces, then $T|_{\text{Ker}(S)}$ is surjective if and only if $\text{Ker}(T) + \text{Ker}(S) = V$. Clearly, this relation is symmetric in S and T , and we obtain the thesis. \square

We give one more introductory result concerning the existence of time-dependent local referentials for vector bundles defined in a neighborhood of a given curve. We need the following definition.

Definition 2.2. Let (\mathcal{M}, \bar{g}) be a Riemannian manifold and $x \in \mathcal{M}$. A positive number $r \in \mathbb{R}^+$ is said to be a *normal radius* for x if $\exp_x : B_r(0) \mapsto B_r(x)$ is a diffeomorphism, where \exp is the exponential map of (\mathcal{M}, \bar{g}) , $B_r(0)$ the open ball of radius r around $0 \in T_x \mathcal{M}$ and $B_r(x)$ is the open ball of radius r around $x \in \mathcal{M}$. We say that r is *totally normal* for x , if r is a normal radius for all $y \in B_r(x)$.

By a simple argument in Riemannian geometry, it is easy to see that if $K \subset \mathcal{M}$ is a compact subset, then there exists $r > 0$ which is totally normal for all $x \in K$.

Given an vector bundle $\pi : \xi \mapsto \mathcal{M}$ of rank k over a manifold \mathcal{M} , a *time-dependent local referential* of ξ is a family of smooth maps $X_i : U \mapsto \xi, i = 1, \dots, k$, defined on an open subset $U \subseteq \mathbb{R} \times \mathcal{M}$ such that $\{X_i(t, x)\}_{i=1}^k$ is a basis of the fiber ξ_x for all $(t, x) \in U$.

Lemma 2.3. *Let \mathcal{M} be a finite dimensional manifold, let $\pi : \xi \mapsto \mathcal{M}$ be a vector bundle over \mathcal{M} and let $\gamma : [a, b] \mapsto \mathcal{M}$ be a continuous curve. Then, there exists an open subset $U \subseteq \mathbb{R} \times \mathcal{M}$ containing the graph of γ and a smooth time-dependent local referential of ξ defined in U .*

Proof. We first consider the case that γ is a smooth curve. Let us choose an arbitrary connection in ξ , an arbitrary Riemannian metric \bar{g} on \mathcal{M} and a smooth extension $\gamma : [a - \varepsilon, b + \varepsilon] \mapsto \mathcal{M}$ of γ with $\varepsilon > 0$. Since the image of γ is compact in \mathcal{M} , there exists $r > 0$ which is a normal radius for all $\gamma(t), t \in [a - \varepsilon, b + \varepsilon]$. We define U to be the open set:

$$U = \{(t, x) \in \mathbb{R} \times \mathcal{M} : t \in]a - \varepsilon, b + \varepsilon[, x \in B_r(\gamma(t))\}.$$

Now let $\bar{X}_1, \dots, \bar{X}_k$ be a referential of ξ along γ ; for instance, this referential can be chosen by parallel transport along γ relative to the connection on ξ . Finally, we obtain a time-dependent local referential for ξ in U by setting for $(t, x) \in U$ and for $i = 1, \dots, k$, $X_i(t, x)$ equal to the parallel transport (relative to the connection of ξ) of $\bar{X}_i(t)$ along the radial geodesic joining $\gamma(t)$ and x .

The general case of a continuous curve is easily obtained by a density argument. Let $\gamma : [a, b] \mapsto \mathcal{M}$ be continuous and let $r > 0$ be a totally normal radius for $\gamma(t)$ for all $t \in [a, b]$. Let $\gamma_1 : [a, b] \mapsto \mathcal{M}$ be any smooth curve such that $\text{dist}(\gamma(t), \gamma_1(t)) < r$ for all t , where dist is the distance induced by the Riemannian metric \bar{g} on \mathcal{M} . Then, if we repeat the above proof for the curve γ_1 , the open set U thus obtained will contain the graph of γ , and we are done. \square

Let us now consider a sub-Riemannian manifold, that is a triple $(\mathcal{M}, \mathcal{D}, g)$ where \mathcal{M} is a smooth n -dimensional manifold, \mathcal{D} a smooth distribution in \mathcal{M} of codimension k and g a smoothly varying positive inner product on \mathcal{D} .

A curve $\gamma : [a, b] \mapsto \mathcal{M}$ is said to be \mathcal{D} -horizontal, or simply horizontal, if it is absolutely continuous and if $\dot{\gamma}(t) \in \mathcal{D}$ for almost all $t \in [a, b]$. As we did in the proof of Lemma 2.3, we will use sometimes auxiliary structures on \mathcal{M} , which are chosen (in a non-canonical way) once for all. We therefore assume that \bar{g} is a given Riemannian metric tensor on \mathcal{M} such that $\bar{g}|_{\mathcal{D}} = g$, that \mathcal{D}_1 is a k -dimensional distribution in \mathcal{M} which is complementary to \mathcal{D} (for instance, \mathcal{D}_1 is the \bar{g} -orthogonal distribution to \mathcal{D}), and we also assume that ∇ is a linear connection in $T\mathcal{M}$ which is adapted to the decomposition $\mathcal{D} \oplus \mathcal{D}_1$, i.e., the covariant derivative of vector fields in \mathcal{D} (resp., in \mathcal{D}_1) belongs to \mathcal{D} (resp., to \mathcal{D}_1). For the construction of these objects, one can consider an arbitrary Riemannian metric \tilde{g} on \mathcal{M} . Then, one defines \mathcal{D}_1 as the \tilde{g} -orthogonal complement of \mathcal{D} and $\bar{g}|_{\mathcal{D}_1} = \tilde{g}|_{\mathcal{D}_1}$; for the connection ∇ , it suffices to choose any pair of connections ∇_0 and ∇_1 , respectively, on the vector bundles \mathcal{D} and \mathcal{D}_1 and then one sets $\nabla = \nabla_0 \oplus \nabla_1$. Observe that the connection ∇ constructed in this way is not torsion free; we denote by τ the torsion of ∇ :

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Using Lemma 2.3, we describe \mathcal{D} locally as the kernel of a time-dependent \mathbb{R}^k -valued 1-form.

Proposition 2.4. *Let $\gamma : [a, b] \mapsto \mathcal{M}$ be a continuous curve. Then, there exists an open subset $U \subseteq \mathbb{R} \times \mathcal{M}$ containing the graph of γ and a smooth time-dependent \mathbb{R}^k -valued*

1-form θ defined in U , with $\theta_{(t,x)} : T_x\mathcal{M} \mapsto \mathbb{R}^k$ a surjective linear map and $\mathcal{D}_x = \text{Ker}(\theta_{(t,x)})$ for all $(t, x) \in U$.

Proof. Let ξ be the subbundle of the cotangent bundle $T\mathcal{M}^*$ given by the annihilator \mathcal{D}^0 of \mathcal{D} . Apply Lemma 2.3 to ξ and set $\theta = (\theta_1, \dots, \theta_k)$, where $\{\theta_i\}_{i=1}^k$ is a time-dependent local referential of ξ defined in an open neighborhood of the graph of γ . \square

Observe that since \mathcal{D}_1 is complementary to \mathcal{D} for all $(t, x) \in U$ the map

$$\theta_{(t,x)} : \mathcal{D}_1 \mapsto \mathbb{R}^k$$

is an isomorphism.

Let us now consider the following spaces of curves in \mathcal{M} .

We denote by $L^2([a, b], \mathbb{R}^m)$ the Hilbert space of Lebesgue square integrable \mathbb{R}^m -valued maps on $[a, b]$ and by $H^1([a, b], \mathbb{R}^m)$ the Sobolev space of all absolutely continuous maps $x : [a, b] \mapsto \mathbb{R}^m$ with derivative in $L^2([a, b], \mathbb{R}^m)$. Finally, we denote by $H^1([a, b], \mathcal{M})$ the set of curves $x : [a, b] \mapsto \mathcal{M}$ such that for any local chart (U, ϕ) on \mathcal{M} , with $\phi : U \subset \mathcal{M} \mapsto \mathbb{R}^n$, and for any closed interval $I \subset x^{-1}(U)$, the map $\phi \circ (x|_I)$ is in $H^1(I, \mathbb{R}^n)$. It is well known that $H^1([a, b], \mathcal{M})$ is an infinite dimensional smooth manifold modeled on the Hilbert space $H^1([a, b], \mathbb{R}^n)$ (see for instance [12] for a recent reference on these issues).

For all pairs of points $q_0, q_1 \in \mathcal{M}$, we define the following sets of curves in \mathcal{M} :

$$\begin{aligned} H_{q_0}^1([a, b], \mathcal{M}) &= \{x \in H^1([a, b], \mathcal{M}) : x(a) = q_0\}, \\ H_{q_0, q_1}^1([a, b], \mathcal{M}) &= \{x \in H^1([a, b], \mathcal{M}) : x(a) = q_0, x(b) = q_1\}, \\ H^1([a, b], \mathcal{D}, \mathcal{M}) &= \{x \in H^1([a, b], \mathcal{M}) : \dot{x}(t) \in \mathcal{D} \text{ a.e. on } [a, b]\}, \\ H_{q_0}^1([a, b], \mathcal{D}, \mathcal{M}) &= H^1([a, b], \mathcal{D}, \mathcal{M}) \cap H_{q_0}^1([a, b], \mathcal{M}), \\ H_{q_0, q_1}^1([a, b], \mathcal{D}, \mathcal{M}) &= H^1([a, b], \mathcal{D}, \mathcal{M}) \cap H_{q_0, q_1}^1([a, b], \mathcal{M}). \end{aligned} \quad (1)$$

We prove that the sets $H_{q_0}^1([a, b], \mathcal{M})$, $H_{q_0, q_1}^1([a, b], \mathcal{M})$, $H^1([a, b], \mathcal{D}, \mathcal{M})$ and $H_{q_0}^1([a, b], \mathcal{D}, \mathcal{M})$, are smooth submanifolds of $H^1([a, b], \mathcal{M})$ for all $q_0, q_1 \in \mathcal{M}$. However, in general, the space $H_{q_0, q_1}^1([a, b], \mathcal{D}, \mathcal{M})$, consisting of horizontal curves joining the two fixed points q_0 and q_1 , is *not* a submanifold of $H_{q_0, q_1}^1([a, b], \mathcal{M})$, and this fact is precisely the origin of difficulties when one tries to develop a variational theory for sub-Riemannian geodesics.

In order to see that $H_{q_0}^1([a, b], \mathcal{M})$ and $H_{q_0, q_1}^1([a, b], \mathcal{M})$ are submanifolds of $H^1([a, b], \mathcal{M})$, simply observe that the map

$$\mathcal{E}_{a,b} : \gamma \mapsto (\gamma(a), \gamma(b))$$

is a submersion of $H^1([a, b], \mathcal{M})$ into $\mathcal{M} \times \mathcal{M}$.

Then, $H_{q_0}^1([a, b], \mathcal{M}) = \mathcal{E}_{a,b}^{-1}(\{q_0\} \times \mathcal{M})$ and $H_{q_0, q_1}^1([a, b], \mathcal{M}) = \mathcal{E}_{a,b}^{-1}(q_0, q_1)$ are smooth submanifolds of $H^1([a, b], \mathcal{M})$.

As to the regularity of $H_{q_0}^1([a, b], \mathcal{D}, \mathcal{M})$, we will now show that this set can be covered by a family of open subset $\{\mathcal{U}_\alpha\}$ of $H_{q_0}^1([a, b], \mathcal{M})$ such that each intersection $H_{q_0}^1([a, b],$

$\mathcal{D}, \mathcal{M}) \cap \mathcal{U}_\alpha$ is the inverse image of a submersion of \mathcal{U}_α in the Hilbert space $L^2([a, b], \mathbb{R}^k)$. The regularity of $H^1([a, b], \mathcal{D}, \mathcal{M})$ will follow by a similar argument.

To this aim, let γ_0 be a fixed curve in $H_{q_0}^1([a, b], \mathcal{M})$ and let $U_{\gamma_0} \subset \mathbb{R} \times \mathcal{M}$ be an open set containing the graph of γ_0 and that is the domain of the map θ of Proposition 2.4. Denote by $H_{q_0}^1([a, b], \mathcal{M}, U_{\gamma_0})$ the open subset of $H_{q_0}^1([a, b], \mathcal{M})$ consisting of those curves whose graphs is contained in U_{γ_0} :

$$H_{q_0}^1([a, b], \mathcal{M}, U_{\gamma_0}) = \{\gamma \in H_{q_0}^1([a, b], \mathcal{M}) : (t, \gamma(t)) \in U_{\gamma_0} \text{ for all } t \in [a, b]\}. \tag{2}$$

Let $\Theta : H_{q_0}^1([a, b], \mathcal{M}, U_{\gamma_0}) \mapsto L^2([a, b], \mathbb{R}^k)$ be the smooth map defined by

$$\Theta(\gamma)(t) = \theta_{(t, \gamma(t))}(\dot{\gamma}(t)). \tag{3}$$

Clearly, $H_{q_0}^1([a, b], \mathcal{M}, U_{\gamma_0}) \cap H_{q_0}^1([a, b], \mathcal{D}, \mathcal{M}) = \Theta^{-1}(0)$.

Proposition 2.5. Θ is a submersion.

Proof. Clearly Θ is smooth because θ is smooth. To compute the differential of Θ , we use the connection ∇ adapted to the decomposition $T\mathcal{M} = \mathcal{D} \oplus \mathcal{D}_1$ introduced above. Let $\gamma \in H_{q_0}^1([a, b], \mathcal{M}, U_{\gamma_0})$ be fixed and let $V \in T_\gamma H_{q_0}^1([a, b], \mathcal{M})$, i.e., V is a vector field of class H^1 along γ with $V(a) = 0$. We write $V = V_{\mathcal{D}} + V_{\mathcal{D}_1}$ with $V_{\mathcal{D}}(t) \in \mathcal{D}$ and $V_{\mathcal{D}_1}(t) \in \mathcal{D}_1$ for all t ; using the properties of ∇ , we compute easily

$$d\Theta(\gamma)[V](t) = [\nabla_V \theta]_{(t, \gamma(t))}(\dot{\gamma}(t)) + \theta_{(t, \gamma(t))}(\nabla_{\dot{\gamma}(t)} V) + \theta_{(t, \gamma(t))}(\tau(V(t), \dot{\gamma}(t))), \tag{4}$$

where $\nabla_V \theta$ is the covariant derivative of $\theta_{(t, \cdot)}$.

Let now $f \in L^2([a, b], \mathbb{R}^k)$ be fixed; for the surjectivity of $d\Theta(\gamma)$, we want to solve the equation in V : $d\Theta(\gamma)[V] = f$. To this aim, we choose $V_{\mathcal{D}_0} = 0$, and we get

$$\theta_{(t, \gamma(t))}(\nabla_{\dot{\gamma}(t)} V_{\mathcal{D}_1}) + [\nabla_{V_{\mathcal{D}_1}} \theta]_{(t, \gamma(t))}(\dot{\gamma}(t)) + \theta_{(t, \gamma(t))}(\tau(V_{\mathcal{D}_1}(t), \dot{\gamma}(t))) = f. \tag{5}$$

Since $\theta_{(t, \gamma(t))} : (\mathcal{D}_1)_{\gamma(t)} \mapsto \mathbb{R}^k$ is an isomorphism, (5) is equivalent to a first order linear differential equation in $V_{\mathcal{D}_1}$, that admits a unique solution satisfying $V_{\mathcal{D}_1}(a) = 0$. Observe that since $\gamma \in H^1([a, b], \mathcal{M})$ by (5), we get that V is also of class H^1 , and we are done. \square

Corollary 2.6. $H^1([a, b], \mathcal{D}, \mathcal{M})$ and $H_{q_0}^1([a, b], \mathcal{D}, \mathcal{M})$ are smooth submanifolds of $H^1([a, b], \mathcal{M})$.

We now consider the *endpoint mapping* end: $H_{q_0}^1([a, b], \mathcal{M}) \mapsto \mathcal{M}$ given by

$$\text{end}(\gamma) = \gamma(b).$$

It is easy to see that end is a submersion, hence we have the following corollary.

Corollary 2.7. Let $\gamma_0 \in H_{q_0}^1([a, b], \mathcal{M})$ be fixed and let $H_{q_0}^1([a, b], \mathcal{M}, U_{\gamma_0}), \Theta$ be defined as in (2) and (3).

Then, for all $\gamma \in \Theta^{-1}(0) \cap \text{end}^{-1}(q_1) = H_{q_0, q_1}^1([a, b], \mathcal{D}, \mathcal{M})$, the restriction $\Theta|_{H_{q_0}^1([a, b], \mathcal{M}, U_{\gamma_0}) \cap H_{q_0, q_1}^1([a, b], \mathcal{M})}$ is a submersion if and only if the restriction $\text{end}|_{H_{q_0}^1([a, b], \mathcal{D}, \mathcal{M})}$ is a submersion.

Proof. It follows immediately from Lemma 2.1 and Proposition 2.5. □

Definition 2.8. A curve $\gamma \in H_{q_0, q_1}^1([a, b], \mathcal{D}, \mathcal{M})$ is said to be *regular* if the restriction $\text{end}|_{H_{q_0}^1([a, b], \mathcal{D}, \mathcal{M})}$ is a submersion at γ . If γ is not regular, then it is called an *abnormal extremal*.

Observe that the notion of abnormal extremality is not related to any sort of extremality with respect to the length or the action functional, but rather to lack of regularity in the geometry of the space of horizontal paths. The smoothness of length minimizing abnormal extremals is an open question.

3. Normal geodesics

In order to define the normal geodesics in a sub-Riemannian manifold, we introduce a Hamiltonian setup in $T\mathcal{M}^*$ as follows.

Let us consider the cotangent bundle $T\mathcal{M}^*$ endowed with its canonical symplectic form ω . Recall that ω is defined by $\omega = -d\vartheta$, ϑ being the canonical 1-form on $T\mathcal{M}^*$ given by $\vartheta_p(\rho) = p(d\pi_p(\rho))$, where $\pi : T\mathcal{M}^* \mapsto \mathcal{M}$ is the projection, $p \in T\mathcal{M}^*$ and $\rho \in T_p T\mathcal{M}^*$. Let $H : T\mathcal{M}^* \mapsto \mathbb{R}$ be a smooth function; we call such a function a *Hamiltonian* in $(T\mathcal{M}^*, \omega)$. The *Hamiltonian vector field* of H is the smooth vector field on $T\mathcal{M}^*$ denoted by \vec{H} and defined by the relation $dH(p) = \omega(\vec{H}(p), \cdot)$; the integral curves of \vec{H} are called the *solutions* of the Hamiltonian H . With a slight abuse of terminology, we will say that a smooth curve $\gamma : [a, b] \mapsto \mathcal{M}$ is a solution of the Hamiltonian H if it admits a lift $\Gamma : [a, b] \mapsto T\mathcal{M}^*$ that is a solution of H .

More in general, one can consider *time-dependent* Hamiltonian functions on $T\mathcal{M}^*$, which are smooth maps defined on an open subset U of $\mathbb{R} \times T\mathcal{M}^*$. In this case, the Hamiltonian flow \vec{H} is a time-dependent vector field in $T\mathcal{M}^*$, and its integral curves in $T\mathcal{M}^*$ are again called the solutions of the Hamiltonian H .

A *symplectic chart* in $T\mathcal{M}^*$ is a local chart taking values in $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ whose differential at each point is a symplectomorphism from the tangent space $T_p(T\mathcal{M}^*)$ to $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ endowed with the canonical symplectic structure. Given a chart $q = (q_1, \dots, q_n)$ in \mathcal{M} , we get a symplectic chart (q, p) on $T\mathcal{M}^*$, where $p = (p_1, \dots, p_n)$ is defined by $p_i(\alpha) = \alpha(\partial/\partial q_i)$. We denote by $\{\partial/\partial q_i, \partial/\partial p_j\}$, $i, j = 1, \dots, n$, the corresponding local referential for $T(T\mathcal{M}^*)$, and by $\{dq_i, dp_j\}$ the local referential of $T(T\mathcal{M}^*)^*$. We have

$$\omega = \sum_{i=1}^n dq_i \wedge dp_i, \quad \vec{H} = \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right).$$

In the symplectic chart (q, p) , a solution $\Gamma(t) = (q(t), p(t))$ of the Hamiltonian H is the solution of the Hamilton equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}. \tag{6}$$

Definition 3.1. A normal geodesic in the sub-Riemannian manifold $(\mathcal{M}, \mathcal{D}, g)$ is a curve $\gamma : [a, b] \mapsto \mathcal{M}$ that admits a lift $\Gamma : [a, b] \mapsto T\mathcal{M}^*$ which is a solution of the sub-Riemannian Hamiltonian $H : T\mathcal{M}^* \mapsto \mathbb{R}$ given by

$$H(p) = \frac{1}{2}g^{-1}(p|_{\mathcal{D}}, p|_{\mathcal{D}}), \tag{7}$$

where g^{-1} is the induced inner product in \mathcal{D}^* . In this case, we say that Γ is a *Hamiltonian lift* of γ .

The Hamilton equations for the sub-Riemannian Hamiltonian (7) will be computed explicitly in Section 5 (formula (31)). It will be seen that the first of the two equations means that the solutions in \mathcal{M} are horizontal curves and that $\Gamma|_{\mathcal{D}} = g(\dot{\gamma}, \cdot)$ (see Remark 5.3).

We remark that a normal geodesic need not be regular in the sense of Definition 2.8, hence there are geodesics that are at the same time normal and abnormal. Observe also that, in general, a normal geodesic γ may admit more than one Hamiltonian lift Γ . This phenomenon occurs precisely when γ is at the same time a normal geodesic and an abnormal extremizer.

4. Abnormal extremals and the endpoint mapping

In this section, we give necessary and sufficient conditions for a curve to be an abnormal extremal in terms of the symplectic structure of the cotangent bundle $T\mathcal{M}^*$. We describe a coordinate system in the Hilbert manifold $H_{q_0}^1([a, b], \mathcal{M})$ which is compatible with the submanifold $H_{q_0}^1([a, b], \mathcal{D}, \mathcal{M})$. This will provide an explicit description of the tangent space $T_{\gamma}H_{q_0}^1([a, b], \mathcal{D}, \mathcal{M})$ which will allow us to compute the image of the differential of the restriction of the endpoint mapping to $H_{q_0}^1([a, b], \mathcal{D}, \mathcal{M})$.

Let \mathcal{M} be a manifold endowed with a distribution \mathcal{D} , with $\dim(\mathcal{M}) = n$ and $\text{codim}(\mathcal{D}) = k$. The sub-Riemannian metric will be irrelevant in the theory of this section. Let $U \subset \mathbb{R} \times \mathcal{M}$ be an open set and let X_1, \dots, X_n be a time-dependent referential of $T\mathcal{M}$ defined in U . We say that such referential is *adapted* to the distribution \mathcal{D} if X_1, \dots, X_{n-k} forms a referential for \mathcal{D} .

It follows easily from Lemma 2.3 that given a continuous curve $\gamma : [a, b] \mapsto \mathcal{M}$, there exists an open set $U \subset \mathbb{R} \times \mathcal{M}$ containing the graph of γ and a referential of $T\mathcal{M}$ defined in U which is adapted to \mathcal{D} . Namely, one chooses a vector subbundle $\mathcal{D}_1 \subset T\mathcal{M}$ such that $T\mathcal{M} = \mathcal{D} \oplus \mathcal{D}_1$ and then apply Lemma 2.3 to both \mathcal{D} and \mathcal{D}_1 .

Given a time-dependent referential of $T\mathcal{M}$ defined in an open set $U \subset \mathbb{R} \times \mathcal{M}$, we are going to associate to it a map

$$\mathcal{B} : H^1([a, b], \mathcal{M}, U) \mapsto L^2([a, b], \mathbb{R}^n),$$

where $H^1([a, b], \mathcal{M}, U)$ denotes the open set in $H^1([a, b], \mathcal{M})$ consisting of curves whose graph is contained in U . We define \mathcal{B} by

$$\mathcal{B}(\gamma) = h, \quad (8)$$

where $h = (h_1, \dots, h_n)$ is given by

$$\dot{\gamma}(t) = \sum_{i=1}^n h_i(t) X_i(t, \gamma(t)) \quad (9)$$

for almost all $t \in [a, b]$. The map \mathcal{B} is smooth. Its differential is computed in the following lemma.

Lemma 4.1. *Let $\gamma \in H^1([a, b], \mathcal{M}, U)$ and v be an H^1 vector field along γ . Set $h = \mathcal{B}(\gamma)$, $z = d\mathcal{B}_\gamma(v)$. We define a time-dependent vector field in U by*

$$X(t, x) = \sum_{i=1}^n h_i(t) X_i(t, x), \quad (t, x) \in U, \quad (10)$$

and a vector field w along γ by

$$w(t) = \sum_{i=1}^n z_i(t) X_i(t, \gamma(t)). \quad (11)$$

Given a chart (q_1, \dots, q_n) defined in an open set $V \subset \mathcal{M}$, denote by $\tilde{v}(t)$, $\tilde{X}(t, q)$ and $\tilde{w}(t)$ the representation in coordinates of v , X and w , respectively. Then, the following relation holds:

$$\frac{d}{dt} \tilde{v}(t) = \frac{\partial \tilde{X}}{\partial q}(t, \gamma(t)) \tilde{v}(t) + \tilde{w}(t) \quad (12)$$

for all $t \in [a, b]$ such that $\gamma(t) \in V$.

Proof. Simply consider a variation of γ with variational vector field v and differentiate relation (9) with respect to the variation parameter, using the local chart. \square

Corollary 4.2. *The restriction of the map \mathcal{B} to the set*

$$H_{q_0}^1([a, b], \mathcal{M}, U) = H_{q_0}^1([a, b], \mathcal{M}) \cap H^1([a, b], \mathcal{M}, U)$$

is a local chart, taking values in an open subset of $L^2([a, b], \mathbb{R}^n)$.

Proof. For $\gamma \in H_{q_0}^1([a, b], \mathcal{M})$ the tangent space $T_\gamma H_{q_0}^1([a, b], \mathcal{M})$ consists of those H^1 vector fields v along γ such that $v(a) = 0$. For a fixed $z \in L^2([a, b], \mathbb{R}^n)$, formula (12) is a first order linear differential equation for \tilde{v} ; Lemma 4.1 and standard results of existence and uniqueness of solutions of linear differential equations imply that the differential of \mathcal{B} at any $\gamma \in H_{q_0}^1([a, b], \mathcal{M}, U)$ maps the tangent space $T_\gamma H_{q_0}^1([a, b], \mathcal{M})$ isomorphically onto

$L^2([a, b], \mathbb{R}^n)$. It follows from the inverse function theorem that \mathcal{B} is a local diffeomorphism in $H_{q_0}^1([a, b], \mathcal{M}, U)$. Finally, by standard results on uniqueness of solutions of differential equations, we see that the restriction of \mathcal{B} to $H_{q_0}^1([a, b], \mathcal{M}, U)$ is injective. \square

If the referential X_1, \dots, X_n defining \mathcal{B} is adapted to \mathcal{D} , then a curve γ in $H_{q_0}^1([a, b], \mathcal{M}, U)$ is horizontal if and only if $\mathcal{B}(\gamma) = h$ satisfies $h_{n-k+1} = \dots = h_n = 0$. This means that \mathcal{B} is a *submanifold chart* for $H_{q_0}^1([a, b], \mathcal{D}, \mathcal{M})$. This observation will provide a good description of the tangent space $T_\gamma H_{q_0}^1([a, b], \mathcal{D}, \mathcal{M})$.

Let $\gamma \in H_{q_0}^1([a, b], \mathcal{M}, U)$ and set $h = \mathcal{B}(\gamma)$. Define a time-dependent vector field X in U as in (10). By Lemma 4.1, the kernel $\text{Ker } d\mathcal{B}_\gamma$ is the vector subspace of $T_\gamma H^1([a, b], \mathcal{M})$ consisting of those v , whose representation in coordinates \tilde{v} satisfy the homogeneous part of the linear differential equation (12), namely

$$\frac{d}{dt} \tilde{v}(t) = \frac{\partial \tilde{X}}{\partial q}(t, \gamma(t)) \tilde{v}(t). \tag{13}$$

By the uniqueness of the solution of a Cauchy problem, it follows that for all $t \in [a, b]$, the evaluation map

$$\text{Ker } d\mathcal{B}_\gamma \ni v \mapsto v(t) \in T_{\gamma(t)}\mathcal{M}$$

is an isomorphism. Therefore, for every $t \in [a, b]$, we can define a linear isomorphism $\Phi_t : T_{\gamma(a)}\mathcal{M} \mapsto T_{\gamma(t)}\mathcal{M}$ by

$$\Phi_t(v(a)) = v(t), \quad v \in \text{Ker } d\mathcal{B}_\gamma. \tag{14}$$

Using the maps Φ_t , we can give a coordinate free description of the differential of \mathcal{B} , based on the “method of variation of constants” for solving non-homogeneous linear differential equations.

Lemma 4.3. *Let $\gamma \in H_{q_0}^1([a, b], \mathcal{M}, U)$ and $v \in T_\gamma H_{q_0}^1([a, b], \mathcal{M})$. Set $h = \mathcal{B}(\gamma)$ and $z = d\mathcal{B}_\gamma(v)$. Define the objects X , w and Φ_t as in (10), (11) and (14), respectively. Then, the following equality holds:*

$$v(t) = \Phi_t \int_a^t \Phi_s^{-1} w(s) ds. \tag{15}$$

Proof. The right-hand side of (15) vanishes at $t = a$, therefore, to conclude the proof, one has to show only that its representation in local coordinates satisfies the differential equation (12). This follows by direct computation, observing that the representation in local coordinates of the maps Φ_t is a solution of the homogeneous linear differential equation (13). \square

Corollary 4.4. *Suppose that the referential X_1, \dots, X_n defining \mathcal{B} is adapted to \mathcal{D} . Let γ be a horizontal curve in $H_{q_0}^1([a, b], \mathcal{M}, U)$. Then, the tangent space $T_\gamma H_{q_0}^1([a, b], \mathcal{D}, \mathcal{M})$*

consists of all vector fields v of the form (15), where w runs over all L^2 horizontal vector fields along γ .

Proof. Follows directly from Lemma 4.3, observing that \mathcal{B} is a submanifold chart for $H_{q_0}^1([a, b], \mathcal{D}, \mathcal{M})$, as remarked earlier. \square

We now relate the differential of the endpoint map with the symplectic structure of $T\mathcal{M}^*$. We denote by $\mathcal{D}^0 \subset T\mathcal{M}^*$ the annihilator of \mathcal{D} . The restriction $\omega|_{\mathcal{D}^0}$ of the canonical symplectic form of $T\mathcal{M}^*$ to \mathcal{D}^0 is in general no longer non-degenerate and its kernel $\text{Ker}(\omega|_{\mathcal{D}^0})(p)$ at a point $p \in \mathcal{D}^0$ may be non-zero. We say that an absolutely continuous curve $\eta : [a, b] \mapsto \mathcal{D}^0$ is a *characteristic curve* for \mathcal{D} if

$$\dot{\eta}(t) \in \text{Ker}(\omega|_{\mathcal{D}^0})(\eta(t))$$

for almost all $t \in [a, b]$.

We take a closer look at the kernel of $\omega|_{\mathcal{D}^0}$. Let Y be a horizontal vector field in an open subset of \mathcal{M} . We associate to it a Hamiltonian function H_Y defined by

$$H_Y(p) = p(Y(x)),$$

where $x = \pi(p)$. We can now compute the ω -orthogonal complement of $T_p\mathcal{D}^0$ in $T_pT\mathcal{M}^*$. Recall that \vec{H}_Y denotes the corresponding Hamiltonian vector field in $T\mathcal{M}^*$.

Lemma 4.5. *Let $p \in T\mathcal{M}^*$ and set $x = \pi(p)$. The ω -orthogonal complement of $T_p\mathcal{D}^0$ in $T_pT\mathcal{M}^*$ is mapped isomorphically by $d\pi_p$ onto \mathcal{D}_x . Moreover, if Y is a horizontal vector field defined in an open neighborhood of x in \mathcal{M} , then $\vec{H}_Y(p)$ is the only vector in the ω -orthogonal complement of $T_p\mathcal{D}^0$ which is mapped by $d\pi_p$ into $Y(x)$.*

Proof. The function H_Y vanishes on \mathcal{D}^0 and therefore $\omega(\vec{H}_Y, \cdot) = dH_Y$ vanishes on $T_p\mathcal{D}^0$. The conclusion follows by observing that since ω is non-degenerate, the ω -orthogonal complement of $T_p\mathcal{D}^0$ in $T_pT\mathcal{M}^*$ has dimension $n - k = \dim(\mathcal{D}_x)$. \square

Corollary 4.6. *The projection of a characteristic curve of \mathcal{D} is automatically horizontal. Moreover, let $\gamma : [a, b] \mapsto \mathcal{M}$ be a horizontal curve, let X_1, \dots, X_n be a time-dependent referential of $T\mathcal{M}$ adapted to \mathcal{D} , defined in an open subset $U \subset \mathbb{R} \times \mathcal{M}$ containing the graph of γ . Define a time-dependent vector field X in U as in (10). Let $\eta : [a, b] \mapsto \mathcal{D}^0$ be a curve with $\pi \circ \eta = \gamma$. Then η is a characteristic curve of \mathcal{D} , if and only if η is an integral curve of \vec{H}_X .*

Proof. For $p \in \mathcal{D}^0$, the kernel of the restriction of ω to $T_p\mathcal{D}^0$ is equal to the intersection of $T_p\mathcal{D}^0$ with the ω -orthogonal complement of $T_p\mathcal{D}^0$ in $T_pT\mathcal{M}^*$. By Lemma 4.5, it follows that the kernel of $\omega|_{\mathcal{D}^0}$ projects by $d\pi$ into \mathcal{D} , and therefore the projection of a characteristic is always horizontal.

For the second part of the statement, observe that for $t \in [a, b]$, $X(t, \cdot)$ is a horizontal vector field in an open neighborhood of $\gamma(t)$ whose value at $\gamma(t)$ is $\dot{\gamma}(t)$. Therefore $\dot{\eta}(t)$ is ω -orthogonal to $T_{\eta(t)}\mathcal{D}^0$ if and only if $\dot{\eta}(t) = \vec{H}_X(\eta(t))$. \square

Corollary 4.7. *Let $\gamma : [a, b] \mapsto \mathcal{M}$ be a horizontal curve and let X_1, \dots, X_n be a time-dependent referential of $T\mathcal{M}$ adapted to \mathcal{D} , defined in an open subset $U \subset \mathbb{R} \times \mathcal{M}$ containing the graph of γ . Let X be defined as in (10). A curve $\eta : [a, b] \mapsto \mathcal{D}^0$ with $\pi \circ \eta = \gamma$ is a characteristic of \mathcal{D} if and only if its representation $\tilde{\eta}(t) \in \mathbb{R}^{n*}$ in any coordinate chart of \mathcal{M} satisfies the following first order homogeneous linear differential equation:*

$$\frac{d}{dt} \tilde{\eta}(t) = -\frac{\partial \tilde{X}}{\partial q}(t, \gamma(t))^* \tilde{\eta}(t), \tag{16}$$

where \tilde{X} is the representation in coordinates of X .

Proof. Simply use Corollary 4.6 and write the Hamilton equations of \vec{H}_X in coordinates. \square

Differential equation (16) is called the *adjoint system* of (13). It is easily seen that $\tilde{\eta}$ is a solution of (16) if and only if $\tilde{\eta}(t)\tilde{v}(t)$ is constant for every solution \tilde{v} of (13). From this observation, we get the following lemma.

Lemma 4.8. *Let $\gamma : [a, b] \mapsto \mathcal{M}$ be a horizontal curve and suppose that the referential X_1, \dots, X_n defining Φ_t in (14) is adapted to \mathcal{D} . Then a curve $\eta : [a, b] \mapsto \mathcal{D}^0$ with $\pi \circ \eta = \gamma$ is a characteristic for \mathcal{D} if and only if $\eta(t) = (\Phi_t^*)^{-1}(\eta(a))$ for every $t \in [a, b]$.*

Proof. By Corollary 4.7 and the observation above we get that η is a characteristic if and only if $\eta(t)v(t)$ is constant for every $v \in \text{Ker } d\mathcal{B}_\gamma$. The conclusion follows. \square

We can finally prove the main theorem of this section.

Theorem 4.9. *The annihilator of the image of the differential of the restriction of the endpoint mapping to $H_{q_0}^1([a, b], \mathcal{D}, \mathcal{M})$ is given by*

$$\begin{aligned} & \text{Im}(d(\text{end}|_{H_{q_0}^1([a, b], \mathcal{D}, \mathcal{M})})(\gamma))^0 \\ & = \{\eta(b) : \eta \text{ is a characteristic for } \mathcal{D} \text{ and } \pi \circ \eta = \gamma\}. \end{aligned} \tag{17}$$

Proof. By Corollary 4.4, we have

$$\begin{aligned} & \text{Im}(d(\text{end}|_{H_{q_0}^1([a, b], \mathcal{D}, \mathcal{M})})(\gamma)) \\ & = \left\{ \Phi_b \int_a^b \Phi_s^{-1} w(s) ds : w \text{ is an } L^2 \text{ horizontal vector field along } \gamma \right\}. \end{aligned} \tag{18}$$

By Lemma 4.8, if η is a characteristic with $\pi \circ \eta = \gamma$ then $\eta(b)$ annihilates the right-hand

side of (18). Namely

$$\begin{aligned} \eta(b) \left(\Phi_b \int_a^b \Phi_s^{-1} w(s) \, ds \right) &= (\Phi_b^*)^{-1}(\eta(a)) \left(\Phi_b \int_a^b \Phi_s^{-1} w(s) \, ds \right) \\ &= \eta(a) \left(\int_a^b \Phi_s^{-1} w(s) \, ds \right) = \int_a^b \eta(a) \Phi_s^{-1} w(s) \, ds \\ &= \int_a^b (\Phi_s^*)^{-1} \eta(a) w(s) \, ds = \int_a^b \eta(s) w(s) \, ds = 0. \end{aligned} \quad (19)$$

We have to prove that if $\eta_0 \in T_{\gamma(b)}\mathcal{M}^*$ annihilates the right-hand side of (18) then there exists a characteristic η with $\pi \circ \eta = \gamma$ and $\eta(b) = \eta_0$.

Define η by $\eta(t) = (\Phi_t^*)^{-1}(\Phi_b^*(\eta_0))$ for all $t \in [a, b]$. By Lemma 4.8, we only have to prove that $\eta([a, b]) \subset \mathcal{D}^0$. Computing as in (19), we see that since η_0 annihilates the right-hand side of (18), then

$$\int_a^b \eta(s) w(s) \, ds = 0$$

for any horizontal L^2 vector field w along γ . The conclusion follows. \square

Corollary 4.10. *The image of the differential of the restriction of the endpoint mapping to $H_{q_0}^1([a, b], \mathcal{D}, \mathcal{M})$ contains $\mathcal{D}_{\gamma(b)}$.*

Proof. By Theorem 4.9, the annihilator of the image of the differential of the restriction of the endpoint mapping to $H_{q_0}^1([a, b], \mathcal{D}, \mathcal{M})$ is contained in the annihilator of $\mathcal{D}_{\gamma(b)}$. The conclusion follows. \square

The next corollary, which is obtained easily from (17), gives a characterization of singular curves in terms of characteristics.

Corollary 4.11. *An H^1 curve $\gamma : [a, b] \mapsto \mathcal{M}$ is singular if and only if it is the projection of a non-zero characteristic of \mathcal{D} .*

Observe that by Lemma 4.8 a characteristic either never vanishes or is identically zero.

5. The normal geodesics as critical points of the action functional

In this section, we prove that the normal geodesics in $(\mathcal{M}, \mathcal{D}, g)$ correspond to the critical points of the sub-Riemannian action functional defined in the space of horizontal curves joining two subsets of \mathcal{M} . To this aim, we need to introduce a Lagrangian formalism that will be related to the Hamiltonian setup described in Section 3 via the Legendre transform.

We consider the sub-Riemannian action functional E_{SR} defined in the space $H^1([a, b], \mathcal{D}, \mathcal{M})$:

$$E_{\text{SR}}(\gamma) = \frac{1}{2} \int_a^b g(\dot{\gamma}, \dot{\gamma}) \, dt. \quad (20)$$

The problem of minimizing the action functional E_{sR} is essentially equivalent to the problem of minimizing length (see Lemma 5.5 and Corollary A.3).

By Corollary 2.7, given $q_0, q_1 \in \mathcal{M}$, the set $H^1_{q_0, q_1}([a, b], \mathcal{D}, \mathcal{M})$ has the structure of a smooth manifold around the regular curves. It is easy to prove that E_{sR} is smooth in any open subset of $H^1_{q_0, q_1}([a, b], \mathcal{D}, \mathcal{M})$ which has the structure of a smooth manifold; such an open set will be called a *regular* subset of $H^1_{q_0, q_1}([a, b], \mathcal{D}, \mathcal{M})$. We will say that a curve $\gamma \in H^1_{q_0, q_1}([a, b], \mathcal{D}, \mathcal{M})$ is a critical point of E_{sR} if it lies in a regular subset of $H^1_{q_0, q_1}([a, b], \mathcal{D}, \mathcal{M})$ and if it is a critical point of the restriction of E_{sR} to this regular subset. The purpose of this section is to prove that the normal geodesics coincide with the critical points of the E_{sR} in $H^1_{q_0, q_1}([a, b], \mathcal{D}, \mathcal{M})$.

To this goal, we will consider an extension E of E_{sR} to the smooth manifold $H^1([a, b], \mathcal{M})$ defined in terms of the Riemannian extension \bar{g} of the sub-Riemannian metric g that was introduced in Section 2:

$$E(\gamma) = \frac{1}{2} \int_a^b \bar{g}(\dot{\gamma}, \dot{\gamma}) dt, \quad \gamma \in H^1([a, b], \mathcal{M}).$$

Let $\gamma \in H^1_{q_0, q_1}([a, b], \mathcal{D}, \mathcal{M})$ be a regular curve and let θ be the map defined in a neighborhood of the graph of γ given in Proposition 2.4. By the method of Lagrange multipliers, we know that γ is a critical point of E_{sR} if and only if there exists $\lambda \in L^2([a, b], \mathbb{R}^n)$ such that γ is a critical point in $H^1_{q_0, q_1}([a, b], \mathcal{M})$ of the action functional

$$E_\lambda(\gamma) = E(\gamma) - \int_a^b \lambda(t) \cdot \theta_{(t, \gamma(t))}(\dot{\gamma}(t)) dt. \tag{21}$$

We will see in the proof of Proposition 5.2 that the Lagrange multiplier λ associated to a critical point of E_{sR} is indeed a smooth map.

E_λ is the action functional of the time-dependent Lagrangian \mathcal{L}_λ defined on an open subset of $T\mathcal{M}$ given by

$$\mathcal{L}_\lambda(t, v) = \frac{1}{2} \bar{g}(v, v) - \lambda(t) \cdot \theta_{(t, m)}(v), \quad v \in T_m \mathcal{M}. \tag{22}$$

The Lagrangian \mathcal{L}_λ is L^1 in the variable t , moreover, for (almost) all $t \in [a, b]$, the map $v \mapsto \mathcal{L}_\lambda(t, v)$ is smooth. Therefore the critical points of E_λ are curves satisfying the Euler–Lagrange equations; in a chart $q = (q_1, \dots, q_n)$, the equations are

$$\frac{\partial \mathcal{L}_\lambda}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}_\lambda}{\partial \dot{q}} = 0. \tag{23}$$

We recall that if $\mathcal{L} : U \subset \mathbb{R} \times T\mathcal{M}$ is a time-dependent Lagrangian defined on an open subset of $\mathbb{R} \times T\mathcal{M}$, the *fiber derivative* of \mathcal{L} is the map $\mathbb{F}\mathcal{L} : U \mapsto \mathbb{R} \times T\mathcal{M}^*$ given by

$$\mathbb{F}\mathcal{L}(t, v) = (t, d(\mathcal{L}|_{U \cap T_{\pi(v)}\mathcal{M}})(v)),$$

where $\pi : T\mathcal{M} \mapsto \mathcal{M}$ is the projection. For $t \in \mathbb{R}$, we denote by U_t the open subset of $T\mathcal{M}$ consisting of those v 's such that $(t, v) \in U$. The Lagrangian \mathcal{L} is said to be *regular* if, for each t , the map $v \mapsto \mathbb{F}\mathcal{L}(t, v)$ is a local diffeomorphism; \mathcal{L} is said to be *hyper-regular*

if $v \mapsto \mathbb{F}\mathcal{L}(t, v)$ is a diffeomorphism between U_t and an open subset of $T\mathcal{M}^*$. Associated to a hyper-regular Lagrangian \mathcal{L} in $U \subset \mathbb{R} \times T\mathcal{M}$, one has a Hamiltonian H defined on the open subset $\mathbb{F}\mathcal{L}(U)$ by the formula

$$H(\mathbb{F}\mathcal{L}(t, v)) = \mathbb{F}L(t, v)v - \mathcal{L}(t, v), \quad (t, v) \in U.$$

This procedure is called the *Legendre transform* (see [1, Chapter 3]). If \mathcal{L} is a hyper-regular Lagrangian and H is the associated Hamiltonian, then the solutions of the Euler–Lagrange equations (23) of \mathcal{L} correspond, via $\mathbb{F}\mathcal{L}$, to the solutions of the Hamilton equations of H , i.e., a smooth curve $\gamma : [a, b] \mapsto \mathcal{M}$ is a solution of (23) if and only if $\Gamma = \mathbb{F}\mathcal{L} \circ (\gamma, \dot{\gamma})$ is a solution of the Hamiltonian H .

Let us show now that this formalism applies to the case of the Lagrangian \mathcal{L}_λ of (22).

Lemma 5.1. *The Lagrangian \mathcal{L}_λ is hyper-regular.*

Proof. From (22), the fiber derivative $\mathbb{F}\mathcal{L}_\lambda$ is easily computed as

$$\mathbb{F}\mathcal{L}_\lambda(t, v) = \bar{g}(v, \cdot) - \lambda(t) \cdot \theta_{(t,m)} \in T_m\mathcal{M}^*. \quad (24)$$

For each $t \in [a, b]$, the map $\mathbb{F}\mathcal{L}_\lambda(t, \cdot) : T_m\mathcal{M} \mapsto T_m\mathcal{M}^*$ is clearly a diffeomorphism, whose inverse is given by

$$T_m\mathcal{M}^* \ni p \mapsto \bar{g}^{-1}(p + \lambda(t) \cdot \theta_{(t,m)}) \in T_m\mathcal{M}. \quad \square \quad (25)$$

We are finally ready to prove the following proposition.

Proposition 5.2. *Let γ be a regular curve in $H_{q_0, q_1}^1([a, b], \mathcal{D}, \mathcal{M})$. Then, γ is a critical point of E_{SR} if and only if it is a normal sub-Riemannian geodesic in $(\mathcal{M}, \mathcal{D}, g)$.*

Proof. A critical point of E_{SR} is a curve satisfying the Euler–Lagrange equations (23) associated to the Lagrangian \mathcal{L}_λ of (22). By Lemma 5.1, \mathcal{L}_λ is hyper-regular, hence the solutions of (23) correspond via $\mathbb{F}\mathcal{L}_\lambda$ to the solutions of the associated Hamiltonian H_λ computed as follows. First, for $v \in T_m\mathcal{M}$, we have

$$\begin{aligned} \mathbb{F}\mathcal{L}_\lambda(t, v)v - \mathcal{L}_\lambda(t, v) &= \bar{g}(v, v) - \lambda(t) \cdot \theta_{(t,m)}(v) - \frac{1}{2}\bar{g}(v, v) + \lambda(t) \cdot \theta_{(t,m)}(v) \\ &= \frac{1}{2}\bar{g}(v, v). \end{aligned}$$

Then, using (25), we compute

$$H_\lambda(t, q, p) = \frac{1}{2}\bar{g}^{-1}(p + \lambda(t) \cdot \theta_{(t,q)}, p + \lambda(t) \cdot \theta_{(t,q)}). \quad (26)$$

To prove Proposition 5.2, we need to show that if γ is an absolutely continuous curve in \mathcal{M} , then γ is horizontal and it is a solution for the Hamilton equations associated to the Hamiltonian H_λ for some λ if and only if it is a solution of the Hamilton equations associated to the sub-Riemannian Hamiltonian H of formula (7).

The Hamilton equations of H_λ are computed as follows:

$$\frac{dq}{dt} = \bar{g}^{-1}(p + \lambda(t) \cdot \theta_{(t,q)}), \quad \frac{dp}{dt} = -\bar{g}^{-1} \left(\lambda(t) \cdot \frac{\partial \theta_{(t,q)}}{\partial q}, p + \lambda(t) \cdot \theta_{(t,q)} \right). \tag{27}$$

From the horizontality of dq/dt , using the first equation of (27), we get

$$(p + \lambda(t) \cdot \theta_{(t,q)})|_{\mathcal{D}_1} = 0,$$

and since $\theta|_{\mathcal{D}_1}$ is an isomorphism, we get an explicit expression for the Lagrange multiplier λ :

$$\lambda(t) = -p(t) \circ [\theta_{(t,q)}|_{\mathcal{D}_1}]^{-1}. \tag{28}$$

Observe that by a standard *boot-strap* argument from (28) it follows easily that λ is smooth.

We now write the Hamilton equations of the sub-Riemannian Hamiltonian and of H_λ using a suitable time-dependent referential X_1, \dots, X_n of $T\mathcal{M}$. The choice of the referential is done as follows. Let $\theta_1, \dots, \theta_k$ be a time-dependent referential of the annihilator $\mathcal{D}^0 = (\mathcal{D}^\perp)^*$ which is orthonormal with respect to \bar{g}^{-1} . For the orthogonality, it suffices to consider any referential of \mathcal{D}^0 and then to orthonormalize it by the method of Gram–Schmidt. Then, let X_{n-k+1}, \dots, X_n be the referential of \mathcal{D}^\perp obtained by dualizing $\theta_1, \dots, \theta_k$. Finally, let X_1, \dots, X_{n-k} be any orthonormal referential of \mathcal{D} , time-dependent or not.

In the referential X_1, \dots, X_n for $i = 1, \dots, n - k$, we have

$$[\theta_{(t,q)}|_{\mathcal{D}_1}]^{-1} \left[\frac{\partial \theta_{(t,q)}}{\partial q} X_i \right] = \sum_{j=1}^k \left[\frac{\partial \theta_j}{\partial q}(t, q) X_i \right] X_{n-k+j}. \tag{29}$$

We can rewrite (27) as

$$\begin{aligned} \frac{dq}{dt} &= \sum_{i=1}^{n-k} p(X_i) X_i + \sum_{i=n-k+1}^n (p(X_i) + \lambda_{i-n+k}) X_i, \\ \frac{dp}{dt} &= -\sum_{i=1}^{n-k} p(X_i) p \left(\frac{\partial X_i}{\partial q} \right) - \sum_{i=n-k+1}^n 2(p(X_i) + \lambda_{i-n+k}) p \left(\frac{\partial X_i}{\partial q} \right), \end{aligned} \tag{30}$$

where $\lambda = (\lambda_1, \dots, \lambda_k)$. On the other hand, the Hamilton equations for H are written as

$$\frac{dq}{dt} = \sum_{i=1}^{n-k} p(X_i) X_i, \quad \frac{dp}{dt} = -\sum_{i=1}^{n-k} p(X_i) p \left(\frac{\partial X_i}{\partial q} \right). \tag{31}$$

Now, if γ is horizontal and it satisfies (30) for some λ it follows that the second sum of the first equation in (30) is zero, and therefore γ satisfies also (31). Conversely, if γ satisfies (31), then γ is horizontal, and defining λ by (28), it is easily seen that γ is a solution of (27). □

Remark 5.3. It follows easily from (31) that if γ is a normal geodesic and Γ is a Hamiltonian lift of γ , then $\Gamma|_{\mathcal{D}} = g(\dot{\gamma}, \cdot)$.

We now consider the case of sub-Riemannian geodesics with endpoints varying in two submanifolds of \mathcal{M} .

Proposition 5.4. *Let $(\mathcal{M}, \mathcal{D}, g)$ be a sub-Riemannian manifold, let $\mathcal{P}, \mathcal{Q} \subset \mathcal{M}$ be smooth submanifolds of \mathcal{M} and assume that \mathcal{Q} is transversal to \mathcal{D} , i.e., $T_q \mathcal{Q} + \mathcal{D}_q = T_q \mathcal{M}$ for all $q \in \mathcal{Q}$. Then, the set*

$$H_{\mathcal{P}, \mathcal{Q}}^1([a, b], \mathcal{D}, \mathcal{M}) = \{x \in H^1([a, b], \mathcal{D}, \mathcal{M}) : x(a) \in \mathcal{P}, x(b) \in \mathcal{Q}\}$$

is a smooth submanifold of $H^1([a, b], \mathcal{M})$. Moreover, the critical points of the sub-Riemannian action functional $E_{\mathbb{S}\mathbb{R}}$ in $H_{\mathcal{P}, \mathcal{Q}}^1([a, b], \mathcal{D}, \mathcal{M})$ are precisely the normal geodesics γ joining \mathcal{P} and \mathcal{Q} that admit a lift $\Gamma : [a, b] \mapsto T\mathcal{M}^$ satisfying the boundary conditions:*

$$\Gamma(a) \in T_{\gamma(a)} \mathcal{P}^0, \quad \Gamma(b) \in T_{\gamma(b)} \mathcal{Q}^0. \quad (32)$$

Proof. The fact that $H_{\mathcal{P}, \mathcal{Q}}^1([a, b], \mathcal{D}, \mathcal{M})$ is a smooth manifold follows easily from the transversality of \mathcal{Q} and Corollary 4.10.

The proof of the second part of the statement is analogous to the proof of Proposition 5.2, keeping in mind that the critical points of the action functional associated to a hyper-regular Lagrangian in the space of curves joining \mathcal{P} and \mathcal{Q} are the solutions of the Hamilton equations whose Hamiltonian lift vanishes on the tangent spaces of \mathcal{P} and \mathcal{Q} . \square

Obviously, the role of \mathcal{P} and \mathcal{Q} in Proposition 5.4 can be interchanged, and the same conclusion holds in the case that \mathcal{P} is transversal to \mathcal{D} .

As a consequence of Proposition 5.4, we get some information on the geometry of sub-Riemannian *balls*. Given a horizontal curve $\gamma : [a, b] \mapsto \mathcal{M}$, we define $\ell(\gamma)$ to be its length:

$$\ell(\gamma) = \int_a^b g(\dot{\gamma}, \dot{\gamma})^{1/2} dt.$$

For $q_0, q_1 \in \mathcal{M}$, we set

$$\text{dist}(q_0, q_1) = \inf\{\ell(\gamma) : \gamma \text{ is a horizontal curve joining } q_0 \text{ and } q_1\} \in [0, +\infty],$$

where such number is infinite if q_0 and q_1 cannot be joined by any horizontal curve. A horizontal curve $\gamma : [a, b] \mapsto \mathcal{M}$ is said to be *length minimizing* between two subsets \mathcal{P} and \mathcal{Q} of \mathcal{M} if $\gamma(a) \in \mathcal{P}$, $\gamma(b) \in \mathcal{Q}$ and

$$\ell(\gamma) = \inf_{\substack{q_0 \in \mathcal{P} \\ q_1 \in \mathcal{Q}}} \text{dist}(q_0, q_1).$$

A horizontal curve γ is said to be *affinely parameterized* if $g(\dot{\gamma}, \dot{\gamma})$ is almost everywhere constant. Every horizontal curve is the reparameterization of an affinely parameterized horizontal curve (see Corollary A.3). Since the sub-Riemannian Hamiltonian is constant on its integral curves, it follows that every normal geodesic is affinely parameterized. Moreover,

using the Hamilton equations (31), it is easy to see that an affine reparameterization of a normal geodesic is again a normal geodesic.

We relate the problem of minimization of the length and of the action functional by the following lemma.

Lemma 5.5. *Let $\gamma : [a, b] \mapsto \mathcal{M}$ be a horizontal curve joining the submanifolds \mathcal{P} and \mathcal{Q} . Then, γ is a minimum of E_{sR} in $H^1_{\mathcal{P}, \mathcal{Q}}([a, b], \mathcal{D}, \mathcal{M})$ if and only if γ is affinely parameterized and γ is a length minimizer between \mathcal{P} and \mathcal{Q} .*

Proof. By Cauchy–Schwartz inequality, we have

$$\ell(\gamma)^2 \leq 2(b - a)E_{\text{sR}}(\gamma)^2,$$

where the equality holds if and only if γ is affinely parameterized. If γ is affinely parameterized and it minimizes length, then for any $\mu \in H^1_{\mathcal{P}, \mathcal{Q}}([a, b], \mathcal{D}, \mathcal{M})$, we have

$$E_{\text{sR}}(\gamma) = \frac{\ell(\gamma)^2}{2(b - a)} \leq \frac{\ell(\mu)^2}{2(b - a)} \leq E_{\text{sR}}(\mu).$$

Hence, γ is a minimum of E_{sR} .

Conversely, suppose that γ is a minimum of E_{sR} . There exists an affinely parameterized horizontal curve $\mu : [a, b] \mapsto \mathcal{M}$ such that γ is a reparameterization of μ (see Corollary A.3). We have

$$E_{\text{sR}}(\gamma) \leq E_{\text{sR}}(\mu) = \frac{\ell(\mu)^2}{2(b - a)} = \frac{\ell(\gamma)^2}{2(b - a)} \leq E_{\text{sR}}(\gamma),$$

hence the above inequalities are indeed equalities, and γ is affinely parameterized.

Now, assume by contradiction that $\rho : [a, b] \mapsto \mathcal{M}$ connects \mathcal{P} and \mathcal{Q} and satisfies $\ell(\rho) < \ell(\gamma)$. By Corollary A.3, we can assume that ρ is affinely parameterized, hence $E_{\text{sR}}(\rho) < E_{\text{sR}}(\gamma)$. This is a contradiction, and we are done. \square

For $q_0 \in \mathcal{M}$ and $r \in \mathbb{R}^+$, the open ball $B_r(q_0)$ is defined by

$$B_r(q_0) = \{q_1 : \text{dist}(q_0, q_1) < r\}.$$

Corollary 5.6. *Suppose that there exists an affinely parameterized length minimizer $\gamma : [a, b] \mapsto \mathcal{M}$ between q_0 and q_1 which is not a normal extremal; set $r = \text{dist}(q_0, q_1)$. Then, any submanifold \mathcal{Q} through q_1 which is transversal to \mathcal{D} at q_1 has non-empty intersection with the open ball $B_r(q_0)$.*

Proof. By contradiction, suppose that we can find a submanifold \mathcal{Q} through q_1 which is transversal to \mathcal{D} at q_1 and disjoint from the open ball $B_r(q_0)$. It follows that γ is a length minimizer between the point q_0 and the submanifold \mathcal{Q} , hence by Lemma 5.5, γ is a minimum point for the action functional in $H^1_{q_0, \mathcal{Q}}([a, b], \mathcal{D}, \mathcal{M})$. By possibly considering a small portion of \mathcal{Q} around q_1 , we can assume that \mathcal{Q} is everywhere transversal to \mathcal{D} . From Proposition 5.4, it follows then that γ is a normal geodesic, which is a contradiction. \square

Remark 5.7. Proposition 5.4 can also be used to establish the *smoothness* of abnormal extremizers, which is in general an open question. Observe indeed that its statement can be rephrased as follows. Let $\gamma : [a, b] \rightarrow \mathcal{M}$ be an affinely parameterized length-minimizer connecting q_0 and q_1 in \mathcal{M} ; set $r = \text{dist}(q_0, q_1)$. If there exists a manifold \mathcal{Q} transverse to \mathcal{D} passing through q_1 which does not intercept the open ball $B(q_0; r)$ then γ is a normal extremal and, consequently, it is smooth.

As a corollary of Proposition 5.2, we also obtain an alternative proof of a result of Liu and Sussmann [9] that gives necessary conditions for length minimizing.

Corollary 5.8. *An affinely parameterized length minimizer is either an abnormal minimizer or a normal geodesic.*

Proof. It follows immediately from Definition 2.8 and Proposition 5.2 and the fact that affinely parameterized length minimizers are minima of the sub-Riemannian action functional. \square

The solutions of sub-Riemannian geodesic problem with variable endpoints in the case that the end-manifold is one-dimensional has a physical interpretation in the context of general relativity (see [5,6]). Such geodesics can be interpreted as the solution of a general relativistic *brachistochrone problem* in a stationary Lorentzian manifold.

Appendix A. Affine parameterization of horizontal curves

In this appendix, we show that every horizontal curve in a sub-Riemannian manifold can be obtained as the reparameterization of an affinely parameterized horizontal curve.

Given two absolutely continuous curves $\gamma : [a, b] \mapsto \mathcal{M}$ and $\mu : [c, d] \mapsto \mathcal{M}$, we say that γ is a *reparameterization* of μ if there exists an absolutely continuous, non-decreasing and surjective map $\sigma : [a, b] \mapsto [c, d]$ such that $\gamma = \mu \circ \sigma$. It can be proven that in this case $\dot{\gamma} = (\dot{\mu} \circ \sigma)\dot{\sigma}$ almost everywhere.

Proposition A.1. *Let (\mathcal{M}, \bar{g}) be a Riemannian manifold, $\gamma : [a, b] \mapsto \mathcal{M}$ an absolutely continuous curve. Then, there exists a unique pair of absolutely continuous maps $\mu : [0, L] \mapsto \mathcal{M}$ and $\sigma : [a, b] \mapsto [0, L]$, with σ non-decreasing and surjective, such that $\bar{g}(\dot{\mu}(t), \dot{\mu}(t)) \equiv 1$ almost everywhere on $[0, L]$ and $\gamma = \mu \circ \sigma$.*

Proof. Suppose that the pair μ, σ satisfying the thesis is found; then we obtain easily

$$\sigma(t) = \ell(\gamma|_{[a,t]}) = \int_a^t \bar{g}(\dot{\gamma}, \dot{\gamma})^{1/2} dt. \quad (\text{A.1})$$

Since σ is surjective, this proves the uniqueness of the pair.

As to the existence, set $L = \ell(\gamma)$ and define σ as in (A.1). Obviously, σ is absolutely continuous, non-decreasing and surjective.

Suppose that $\sigma(s) = \sigma(t)$ for some $s, t \in [a, b]$ with $s < t$. Then, $\ell(\gamma|_{[s,t]}) = 0$, and therefore $\gamma(s) = \gamma(t)$. It follows that there exists a function $\mu : [0, L] \mapsto \mathcal{M}$ with $\mu \circ \sigma = \gamma$. The curve μ is Lipschitz continuous, hence absolutely continuous; for, if $s, t \in [0, L]$, let $s_1, t_1 \in [a, b]$ be such that $\sigma(s_1) = s$ and $\sigma(t_1) = t$. Then

$$\text{dist}(\mu(s), \mu(t)) = \text{dist}(\gamma(s_1), \gamma(t_1)) \leq \ell(\gamma|_{[s_1,t_1]}) = |\sigma(s_1) - \sigma(t_1)| = |s - t|.$$

We are left with the proof that $\bar{g}(\dot{\mu}, \dot{\mu}) \equiv 1$ almost everywhere. To see this, let $t \in [0, L]$ be chosen and let $t_1 \in [a, b]$ be such that $t = \sigma(t_1)$. Then, we have

$$\int_0^t \bar{g}(\dot{\mu}, \dot{\mu})^{1/2} dr = \ell(\mu|_{[0,t]}) = \ell(\gamma|_{[a,t_1]}) = \sigma(t_1) = t. \tag{A.2}$$

The conclusion follows by differentiating (A.2) with respect to t . □

Lemma A.2. *Let \mathcal{M} be a smooth manifold and $\mathcal{D} \subset T\mathcal{M}$ be a smooth distribution. Let $\mu : [a, b] \mapsto \mathcal{M}$ be an absolutely continuous curve; if μ admits a reparameterization which is horizontal, then μ is horizontal.*

Proof. Let $\sigma : [c, d] \mapsto [a, b]$ be an absolutely continuous non-decreasing surjective map with $\gamma = \mu \circ \sigma$ horizontal. Define

$$\begin{aligned} X &= \{t \in [c, d] : \text{the equality } \dot{\gamma}(t) = \dot{\mu}(\sigma(t))\dot{\sigma}(t) \text{ fails to hold}\}, \\ Y &= \{t \in [c, d] : \dot{\sigma}(t) = 0\}. \end{aligned}$$

Clearly, μ is horizontal outside $\sigma(X \cup Y)$; to conclude the proof it suffices to show that $\sigma(X \cup Y)$ has null measure. To see this, observe that X has null measure and therefore $\sigma(X)$ has null measure. Moreover, since $\dot{\sigma} = 0$ in Y , it is not difficult to show that $\sigma(Y)$ has null measure, and we are done. □

Corollary A.3. *Let $(\mathcal{M}, \mathcal{D}, g)$ be a sub-Riemannian manifold and γ a horizontal curve in \mathcal{M} . Then, γ is the reparameterization of a unique horizontal curve $\mu : [0, L] \mapsto \mathcal{M}$ such that $g(\dot{\mu}, \dot{\mu}) \equiv 1$ almost everywhere.*

Proof. Let \bar{g} be any Riemannian extension of g and apply Proposition A.1. The curve μ thus obtained is horizontal by Lemma A.2. □

Appendix B. Local minimality of normal geodesics

The aim of this section is to prove that a sufficiently small segment of a sub-Riemannian normal geodesic is a distance minimizer between an initial submanifold and a point. We will simply adapt the proof of local optimality presented in [9, Appendix C].

Proposition B.1. *Let $(\mathcal{M}, \mathcal{D}, g)$ be a sub-Riemannian manifold, $\mathcal{P} \subset \mathcal{M}$ a submanifold and $\gamma : [a, b] \mapsto \mathcal{M}$ a normal geodesic with $\gamma(a) \in \mathcal{P}$ and such that there exists a*

Hamiltonian lift $\Gamma : [a, b] \mapsto T\mathcal{M}^*$ of γ with $\Gamma(a)|_{T_{\gamma(a)}\mathcal{P}} = 0$. Then, for $\varepsilon > 0$ small enough, $\gamma|_{[a, a+\varepsilon]}$ is a length minimizer between \mathcal{P} and $\gamma(a + \varepsilon)$.

Proof. We can assume without loss of generality that $g(\dot{\gamma}, \dot{\gamma}) = 1$. Let $\mathcal{S} \subset \mathcal{M}$ be a codimension 1 submanifold containing a neighborhood of $\gamma(a)$ in \mathcal{P} and such that $\Gamma(a)|_{T_{\gamma(a)}\mathcal{S}} = 0$. The existence of such a submanifold is easily proved using a coordinate system in \mathcal{M} adapted to \mathcal{P} around $\gamma(a)$. Observe that by Remark 5.3, we have $g^{-1}(\Gamma(a)|_{\mathcal{D}}, \Gamma(a)|_{\mathcal{D}}) = 1$.

Let $\lambda : \mathcal{S} \mapsto T\mathcal{M}^*$ be a 1-form in \mathcal{M} along \mathcal{S} such that $\lambda(x)|_{T_x\mathcal{S}} = 0$, $g^{-1}(\lambda(x)|_{\mathcal{D}}, \lambda(x)|_{\mathcal{D}}) = 1$ for all $x \in \mathcal{S}$ and such that $\lambda(\dot{\gamma}(a)) = \Gamma(a)$. Let $U \subset \mathcal{S}$ be a sufficiently small open subset containing $\gamma(a)$ and let $\varepsilon > 0$ be sufficiently small. Consider the map $\Phi :]a - \varepsilon, a + \varepsilon[\times U \mapsto T\mathcal{M}^*$ such that $t \mapsto \Phi(t, x)$ is a solution of the sub-Riemannian Hamiltonian H defined in (7) and $\Phi(a, x) = \lambda(x)$ for all $x \in U$. Let $F = \pi \circ \Phi$, where $\pi : T\mathcal{M}^* \mapsto \mathcal{M}$ is the projection.

By Remark 5.3, $\Gamma(a)(\dot{\gamma}(a)) = 1$, which implies that $T_{\gamma(a)}\mathcal{M} = T_{\gamma(a)}\mathcal{S} \oplus (\mathbb{R}\dot{\gamma}(a))$. It follows easily that the differential of F at $(a, \gamma(a))$ is an isomorphism, and by the inverse function theorem by possibly passing to smaller ε and U , F is a diffeomorphism between $]a - \varepsilon, a + \varepsilon[\times U$ and an open neighborhood V of $\gamma(a)$ in \mathcal{M} . By possibly taking a smaller V , we can assume that $V \cap \mathcal{P} \subset \mathcal{S}$.

We define a vector field X , a 1-form λ and a smooth map τ on V by setting

$$\tau(F(t, x)) = t, \quad X(F(t, x)) = \frac{d}{dt}F(t, x), \quad \lambda(F(t, x)) = \Phi(t, x)$$

for all $(t, x) \in]a - \varepsilon, a + \varepsilon[\times U$. Since $H \circ \Phi$ does not depend on t , it follows easily that

$$g^{-1}(\lambda|_{\mathcal{D}}, \lambda|_{\mathcal{D}}) = 1. \quad (\text{B.1})$$

We prove next that $\lambda = d\tau$. To this aim, let Ψ_X denote the flow of X , defined on an open subset of $\mathbb{R} \times V$; for $s \in \mathbb{R}$, we set $\Psi_X^s = \Psi_X(s, \cdot)$. Clearly, $t \mapsto F(t, x)$ is an integral curve of X , and therefore we have $\tau \circ \Psi_X^s = s + \tau$, hence $d\tau$ is invariant by the flow of X , i.e.,

$$(\Psi_X^s)^*(d\tau) = d\tau.$$

We show that λ is also invariant by the flow of X ; the equality $\lambda = d\tau$ will follow from the fact that these two 1-forms coincide on \mathcal{S} . For the invariance of λ , we argue as follows: let $x \in U$, $v_0 \in T_x\mathcal{M}$ and $v(t) = d\Psi_X^{t-a}(x)[v_0]$; it suffices to prove that $\lambda(F(t, x))(v(t))$ is constant in t .

In local coordinates $q = (q_1, \dots, q_n)$, v satisfies the following linear differential equation:

$$\frac{dv}{dt} = \frac{\partial X}{\partial q}(v). \quad (\text{B.2})$$

For $t \in]- \varepsilon, \varepsilon[$ fixed, let X_1, \dots, X_{n-k} be an orthonormal frame for \mathcal{D} around $F(t, x)$; by

Remark 5.3, we have $\Phi(t, x)|_{\mathcal{D}} = g(X(F(t, x)), \cdot)$ from which it follows:

$$X = \sum_{i=1}^{n-k} \lambda(X_i) X_i. \tag{B.3}$$

From (B.1), it follows that $\sum_i \lambda(X_i)^2 = 1$, and differentiating this expression, we obtain

$$\sum_{i=1}^{n-k} \lambda(X_i) \frac{\partial}{\partial q} (\lambda(X_i)) = 0. \tag{B.4}$$

From (B.3) and (B.4), it follows:

$$\lambda \left(\frac{\partial X}{\partial q} \right) = \sum_{i=1}^{n-k} \lambda(X_i) \lambda \left(\frac{\partial X_i}{\partial q} \right). \tag{B.5}$$

Using the second Hamilton equation in (31), we finally get

$$\frac{d}{dt} \lambda(F(t, x)) = -\lambda \left(\frac{\partial X}{\partial q} \right). \tag{B.6}$$

Using (B.2) and (B.6) it is easily seen that $\lambda(F(t, x))v(t)$ is constant in t , and λ is invariant by the flow of X .

The equality $\lambda = d\tau$ is thus proven, and by (B.1), we obtain

$$g^{-1}(d\tau|_{\mathcal{D}}, d\tau|_{\mathcal{D}}) = 1. \tag{B.7}$$

Let now $\mu : [a, a + \varepsilon] \mapsto V$ be a horizontal curve with $\mu(a) \in \mathcal{P}$ and $\mu(a + \varepsilon) = \gamma(a + \varepsilon)$. Using (B.7), the length of μ is estimated as follows:

$$\begin{aligned} L(\mu) &= \int_a^{a+\varepsilon} \|\dot{\mu}\| dt \geq \int_a^{a+\varepsilon} d\tau(\dot{\mu}(t)) dt \\ &= \tau(\mu(a + \varepsilon)) - \tau(\mu(a)) = \varepsilon = L(\gamma|_{[a, a+\varepsilon]}). \end{aligned}$$

This implies that $\gamma|_{[a, a+\varepsilon]}$ is a length minimizer between \mathcal{P} and $\gamma(a + \varepsilon)$ among all the horizontal curves with image in V . The conclusion of the proof will follow from the next lemma by possibly considering a smaller ε . □

Lemma B.2. *Let $(\mathcal{M}, \mathcal{D}, g)$ be a sub-Riemannian manifold and let $V \subset \mathcal{M}$ be an open subset. Given $x \in U$ there exists $r > 0$ such that every horizontal curve $\mu : [a, b] \mapsto \mathcal{M}$ with $\mu(a) = x$ and $L(\mu) < r$ satisfies $\mu([a, b]) \subset V$.*

Proof. We compare the sub-Riemannian metric g with the Euclidean metric relative to an arbitrary coordinate system around x . Let $\varphi : W \mapsto \tilde{W}$ be a coordinate system in \mathcal{M} with $x \in W$, $W \subset V$ and \tilde{W} is an open neighborhood of 0 in \mathbb{R}^n . Let $B \subset W$ be the inverse image through φ of a closed ball of radius s , $B[\varphi(x); s] \subset \tilde{W}$. For $m \in W$ and $v \in T_m \mathcal{M}$, denote by $\|v\|_e$ the Euclidean norm of the vector $d\varphi(m)[v]$. The set of vectors $v \in \mathcal{D}$ that are tangent to the points of B with $\|v\|_e = 1$ form a compact subset of $T\mathcal{M}$, in which the

continuous function $v \mapsto g(v, v)^{1/2} = \|v\|$ attains a positive minimum k . Observe that for all $v \in \mathcal{D}$ tangent to some point of B , it is $\|v\| \geq k \cdot \|v\|_e$.

Take $r = ks > 0$. If $\mu : [a, b] \mapsto \mathcal{M}$ is a horizontal curve with $\mu(a) = x$ and $\mu([a, b]) \not\subseteq V$, then there exists $c \in]a, b[$ with $\mu([a, c]) \subset B$ and $\gamma(c) \in \partial B$. Therefore,

$$L(\mu) \geq L(\mu|_{[a,c]}) \geq kL_e(\varphi \circ \mu|_{[a,c]}) \geq ks = r,$$

where L_e denotes the Euclidean length of a curve. This concludes the proof. \square

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